Generators of Hecke Ring for Gelfand Pair (S(2n), B(n))

Kürşat Aker, Feza Gürsey Institute - M. B. Can, Tulane Uni.

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Center of the group ring as a vector space

To study a given group G, we *can* study its *irreducible*

- representations and
- characters.

Center of the group ring plays a fundamental role.

- Character table of G is the change of basis matrix from irreducible characters to conjugacy class sums - two basis of the center of group ring.

Many properties of the group G can be read from its character table.

What about the center of $\mathbb{Z}G$ as a ring?

Theorem (Farahat-Higman, 1959)

Let $\mathcal{Z}_n$ be the the center of integral symmetric group ring, $\mathbb{Z}S(n)$. Then, the ring $\mathcal{Z}_n$ is generated by the elements

$$Z_i := \sum_{w \text{ has exactly } i \text{ cycles}} w \text{ for } i = 1, \dots, n.$$

Cycle Decomposition

- $(1,3)(5,2,6,4) \in S(6)$ is of cycle type $(4,2)$ and has **2** cycles,
- $(1,3)(5,2,6,4) \in S(7)$ is of cycle type $(4,2,1)$ and has **3** cycles, since (1,3)(5,2,6,4)=(1,3)(5,2,6,4)(7).

CONSEQUENCES

- Nakayama's conjecture

  Let p be a prime number. Given two partitions λ and μ of n, then $\chi^\lambda \equiv \chi^\mu \pmod{p}$ if and only if λ, μ have the same p-core.

Later developments

- Jucys, 1974: $Z_i = e_{n-i}(J_1, J_2, \ldots, J_n)$ for $n = 1, 2, \ldots, n$.

  $J_k := \sum$ transpositions in $S(k) - \sum$ transpositions in $S(k-1)$.

- Murphy, 1984:

  Young's Seminormal representations using YJM elements

- Vershik-Okounkov, 1995:

  Representation theory of $S(n)$ by induction on n

Farahat-Higman

Idea of the proof:

Let n go to **infinity** and get rid off all **accidental** relations.

Center → Hecke: Gelfand Pairs

Definition (Gelfand Pairs)

A pair (G, K) is called a *Gelfand pair*

- if 1_K^G is multiplicity-free,
- or equivalently if $End_G(1_K^G)$ is commutative.

We call the latter algebra *Hecke* algebra.

Example (All groups are Gelfand Pairs!)

The pair $(G \times G, \Delta G)$ is Gelfand:

- $1_{\Delta G}^{G\times G} \cong \oplus_{V_i} V_i^* \otimes V_i$,
- $End_{G\times G}(1_{\Delta G}^{G\times G}) \cong \mathcal{Z}(\mathbb{Z}G)$.

$(S(2n), B(n))$ IS A GELFAND PAIR:

- $S(2n)$, symmetric group on $2n$ letters,
- $B(n)$, centralizer of the element

$$t := (1,2)(3,4)\cdots(2n-1,2n).$$

- $1_{B(n)}^{S(2n)} \cong \oplus_{\lambda \vdash n} S^{2\lambda}$, where $2\lambda = (2\lambda_1, 2\lambda_2, 2\lambda_3, \ldots)$,
- Set $\mathcal{H}_n := End_{S(2n)}(1_{B(n)}^{S(2n)})$.

Groups to Gelfand Pairs: A Dictionary

Replace

- Conjugacy classes (sums) by double cosets (sums),
- Irreducible characters by spherical functions.

MULTIPLICATION IN THE CENTER OF $\mathbb{Z}G$

Given two conjugacy classes/class sums C_λ, C_μ in $\mathcal{Z}(\mathbb{Z}G)$, the product $C_\lambda C_\mu$ has the following expansion:

$$C_\lambda C_\mu = \sum_\nu a^\nu_{\lambda\mu} C_\nu,$$

where, C_ν runs over all possible conjugacy classes.
Fix any element $z \in C_\nu$, then

$$a^\nu_{\lambda\mu} := \{(x,y) | xy = z, x \in C_\lambda, y \in C_\mu\}.$$

Multiplication in a Hecke ring

Given two double coset sums K_λ, K_μ in a Hecke ring, the product $K_\lambda K_\mu$ has the following expansion:

$$K_\lambda K_\mu = \sum_\nu b^\nu_{\lambda\mu} K_\nu,$$

where, K_ν runs over all possible double cosets.
Fix any element $z \in K_\nu$, then

$$b^\nu_{\lambda\mu} := \{(x, y) | xy = z, x \in K_\lambda, y \in K_\mu\}.$$

INVARIANTS OF $w \in S(2n)$ WITH RESPECT TO $B(n)$

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 1 & 7 & 2 & 8 & 6 & 4 & 10 & 3 & 9 & 5 & 12 & 11 \end{pmatrix}$$

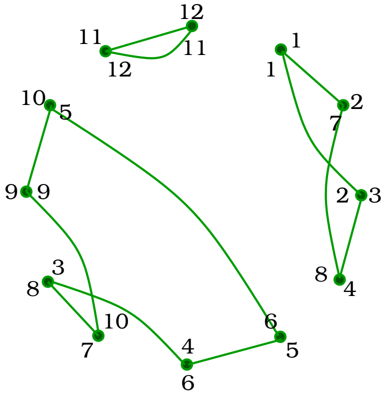

Figure: Coset Type $(3, 2, 1)$ with 3 circuits

AN ANALOGUE OF FH FOR $(S(2n), B(n))$

Theorem (A. - Can, 2010)

The ring $\mathcal{H}_n$ is generated by the elements

$$H_i := \sum_{w \text{ has exactly } i \text{ circuits}} w \text{ for } i = 1, \ldots, n.$$

In terms of YJM elements

Recall that Jucys proved that

$$Z_i = e_{n-i}(J_1, J_2, \dots, J_n)$$

for $n = 1, 2, \dots, n$.
Recently, Zinn-Justin and Matsumoto proved that

$$H_i = e_{n-i}(J_1, J_3, \dots, J_{2n-1})e_{B(n)}$$

for $n = 1, 2, \dots, n$.