Parabolic triple factorisations and their associated geometries

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(joint work with Cheryl E. Praeger and John Bamberg)

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Triple factorisations

Definition

For a finite group G, a triple $\mathcal{T} = (G, A, B)$ is called a *triple factorisation* if $G = ABA$, where $A, B \leq G$.

- $G = AB$ or BA: $\mathcal{T}$ is a degenerate triple factorisation.
- $G \neq AB$: $\mathcal{T}$ is a nondegenerate triple factorisation.

A group with triple factorisation $\mathcal{T} = (G, A, B)$ is sometimes called an *ABA*-**group**.

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TF:=triple factorisation

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Why?

Lie theory

BN-pairs: If G has a BN-pair $\Rightarrow G = BNB$ (Bruhat decomposition)
e.g., Chevalley groups and Twisted groups.

Abstract group theory

For $G = ABA$, study group theoretic properties of G from group theoretic properties of A and B.
e.g. Gorenstein-Herstein (1959): A and B with $\gcd(|A|, |B|) = 1$ $\Rightarrow G$ is solvable.

Geometry

Higman-McLaughlin (1961): every G-flag-transitive rank 2 geometry gives $G = ABA \Leftrightarrow$

Collinearity property: each pair of points lies on at least on line.

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Rank 2 geometries

Notation and Definitions

Suppose that $X = \mathbb{P} \cup \mathbb{L}$ (disjoint union) with

- $\mathbb{P}$: point set;
- $\mathbb{L}$: line set;
- **Incidence relation** $*$ on X: symmetric and reflexive

$$x \text{ and } y \text{ are incident} \Leftrightarrow x * y, \text{ for } x, y \in X.$$

- **flag**: an incident pair (p, ℓ) of π.

Rank 2 geometry: A triple $\pi := (\mathbb{P}, \mathbb{L}, *)$ where

1. two distinct elements of the same type are not incident;
2. each point lies on a line.

The dual of $\pi = (\mathbb{P}, \mathbb{L}, *)$: $\pi^{\vee} = (\mathbb{L}, \mathbb{P}, *)$.

Here, every rank 2 geometry satisfies

- $|\mathbb{P}|$ and $|\mathbb{L}|$ are finite and of size at least 2;
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Coset geometries

Let G be a group with A and B subgroups. Set

- $\mathbb{P} := \{Ax \mid x \in G\}$;
- $\mathbb{L} := \{Bx \mid x \in G\}$;
- $*$ is nonempty intersection:

$$Ax * By \Leftrightarrow Ax \cap By \neq \varnothing$$

Then $(\mathbb{P}, \mathbb{L}, *)$ is a rank 2 geometry called **coset geometry** and denoted by $\mathrm{Cos}(\mathbf{G}; \mathbf{A}, \mathbf{B})$.

Example

$G = \langle x, y \rangle \cong Z_4 \times Z_2$,
$A = \langle x^2 \rangle$ and $B = \langle y \rangle$, where
$x := (1, 2, 3, 8)(4, 5, 6, 7)$ and
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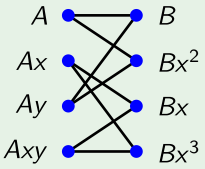

Rank 2 geometries

Flag-transitive geometries

Let $\pi := (\mathbb{P}, \mathbb{L}, *)$ be rank 2 geometry, and set $X := \mathbb{P} \cup \mathbb{L}$.

- An automorphism g of π: a bijection $g : X \to X$ taking points to points, lines to lines and preserving incidence:
$$p * \ell \quad \Leftrightarrow \quad (p)g * (\ell)g.$$
- $\mathrm{Aut}(\pi) := \{g \mid g \text{ is an automorphism of } \pi\}$.
- $G \leq \mathrm{Aut}(\pi)$ acts on points and lines, and so on flags:
$$(p, \ell)^g = ((p)g, (\ell)g), \ (p \in \mathbb{P} \text{ and } \ell \in \mathbb{L}).$$
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Cos($G; A, B$) is G- flag-transitive via
$$(Ax, By)^g := (Axg, Byg),$$
for all $g \in G$, $Ax \in \mathbb{P}$, $By \in \mathbb{L}$.

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Proposition

Let π be rank 2 geometry and $G \leq \mathrm{Aut}(\pi)$. Then π is G-flag transitive $\Leftrightarrow$ $\pi \cong Cos(G; A, B)$ for some subgroups A and B.

For a flag (p, ℓ) of π, $A := G_p$ and $B := G_\ell$

$$\pi \cong Cos(G; G_p, G_\ell)$$

Triple factorisations and rank 2 geometries

Remark

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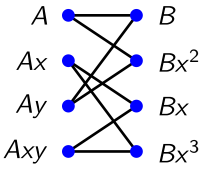

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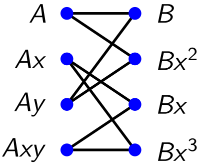

Question 2

Under which conditions a G-flag-transitive rank 2 geometry gives rise to a TF for G?

Collinearly and concurrently connected spaces

Collinearly connected π: each pair of points lies on at least one line.

A collinearly connected space

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Concurrently connected π: each pair of lines meets in at least one point.

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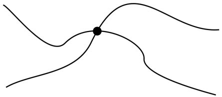

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π is collinearly connected $\Leftrightarrow$ $\pi^{\vee}$ is concurrently connected.

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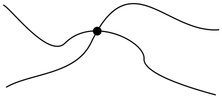

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Example

- All $2-(v,k,\lambda)$ designs are collinearly connected as $\lambda \geq 1$;
- Symmetric designs are both collinearly and concurrently connected.
- Projective spaces $\mathrm{PG}(n-1,q)$ for $n \geq 4$ are collinearly but not concurrently connected: $V := \langle e_1, e_2, e_3, e_4, \ldots, e_n \rangle$, then two lines $\langle e_1, e_2 \rangle$ and $\langle e_3, e_4 \rangle$ do not meet.

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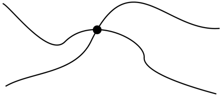

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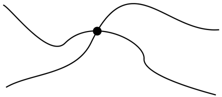

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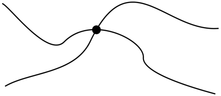

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Example

The Cos$(G; A, B)$ below is neither collinearly, nor concurrently connected:

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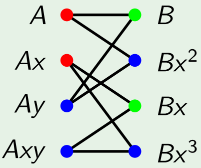

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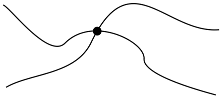

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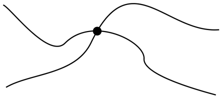

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Higman-McLaughlin Criterion (1961)

1. $\mathrm{Cos}(G; A, B)$ is collinearly connected if and only if $G = ABA$;
2. $\mathrm{Cos}(G; A, B)$ is concurrently connected if and only if $G = BAB$.

Linear spaces

Higman-McLaughlin (1961)

The following are equivalent:

- G is a Geometric ABA-group ($G = ABA$, $A \not\subseteq B$, $B \not\subseteq A$, $AB \cap BA = A \cup B$);
- $\mathrm{Cos}(G; A, B)$ is a (G-flag transitive) linear space.

If G is a **Geometric** ABA**-group**, then G is **primitive** on right cosets of A: A is maximal.

Question

For a given TF $\mathcal{T} = (G, A, B)$, is there any reduction pathway to the case where A is maximal? **YES** (AP-2009)

Parabolic triple factorisations of $\mathrm{GL}(V)$

Let $G := \mathrm{GL}(V)$. Consider the **Grassmannian** set $\mathrm{Gr}_m(V)$ of all m-subspaces of V.

- For $U \in \mathrm{Gr}_m(V)$, the stabiliser subgroup $H := G_U$ of G is a (maximal) **parabolic subgroup** of G.
- A triple factorisation (G, A, B) with A and B parabolic subgroups is called a **parabolic triple factorisation**.

Theorem

Let $G = \mathrm{GL}(V)$, $A := G_U$ and $B := G_W$ with $U \in \mathrm{Gr}_m(V)$ and $W \in \mathrm{Gr}_k(V)$, and let $j := \dim(U \cap W)$. Then

$$G = ABA \Leftrightarrow j \leq \frac{k}{2} + \max\left\{0, m - \frac{n}{2}\right\}.$$

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(m, k, j)-projective spaces

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Let V be a v.s. over a field $\mathbb{F}$, and let $1 \leq m, k < n$ be positive integers. Let j be positive integer satisfying

$$\max\{0, m+k-n\} \leq j \leq \min\{m, k\}.$$

- $\mathbb{P} := \mathrm{Gr}_m(V)$;
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- $(\mathbb{P}, \mathbb{L}, *)$ is a rank 2 geometry called **(m, k, j)-projective space** of V and denoted by $\mathrm{Proj}^j_{(m,k)}(V)$ or $\mathrm{Proj}^j_{(m,k)}(n, \mathbb{F})$.

(m, k, j)-projective spaces

Notation and Definitions

Let V be a v.s. over a field $\mathbb{F}$, and let $1 \leq m, k < n$ be positive integers. Let j be positive integer satisfying

$$\max\{0, m + k - n\} \leq j \leq \min\{m, k\}.$$

- $\mathbb{P} := \mathrm{Gr}_m(V)$;
- $\mathbb{L} := \mathrm{Gr}_k(V)$ (if $m = k$, take $\mathbb{L}$ a copy of $\mathbb{P}$)
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Link to projective geometry

If $j_0 = \min\{m, k\} \Rightarrow *^{j_0}$ is the 'symmetrised inclusion'.

Link to parabolic triple factorisations

- $G := \mathrm{GL}(V)$ acts transitively on flags of $\mathrm{Proj}^j_{(m,k)}(V)$ by
$$(U, W)^g := ((U)g, (W)g).$$
- For a flag (U, W),
$$\mathrm{Proj}^j_{(m,k)}(V) \cong \mathrm{Cos}(G; G_U, G_W),$$
where $A := G_U$ and $B := G_W$ are maximal parabolic.
- $\mathrm{Proj}^j_{(m,k)}(V)$ is collinearly (concurrently) connected $\Leftrightarrow$ $G = ABA$ ($G = BAB$) where $A := G_U$ and $B := G_W$ are parabolic.

Theorem (Alavi-Bamberg-Praeger)

$\mathrm{Proj}^j_{(m,k)}(V)$ is collinearly connected $\Leftrightarrow j \leq \frac{k}{2} + \max\{0, m - \frac{n}{2}\}$.

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(m,k,j)-projective spaces

Collinearity property

- For each (m,k), there exists possible j such that $\mathrm{Proj}^{j}_{(m,k)}(V)$ is collinearly connected.
- There exist parabolic subgroups A and B such that $G = ABA$

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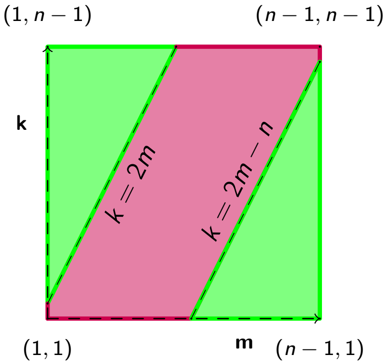


For each $(m, k) \in$ ■ (green), $\forall$ possible j, $\mathrm{Proj}^{j}_{(m,k)}(V)$ is collinearly conn.

For each $(m, k) \in$ ■ (red), $\exists$ possible j_1, j_2 s.t.
$\mathrm{Proj}^{j_1}_{(m,k)}(V)$ is collinearly conn.
$\mathrm{Proj}^{j_2}_{(m,k)}(V)$ is not collinearly conn.

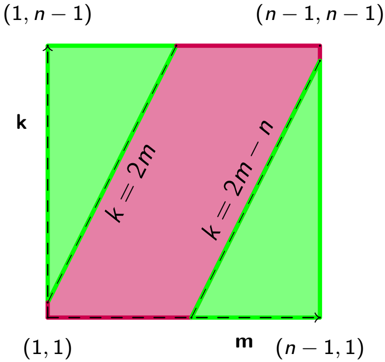
(m,k,j)-projective spaces

collinearly and/or concurrently connected

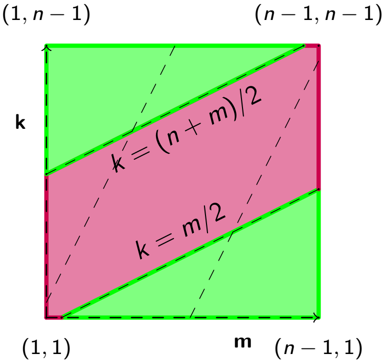
Question 3

Under which conditions $\mathrm{Proj}^{j}_{(m,k)}(V)$ is a collinearly **and/or** concurrently connected space? ($G = ABA$ and/or $G = BAB$)

Collinearity property



Concurrency property



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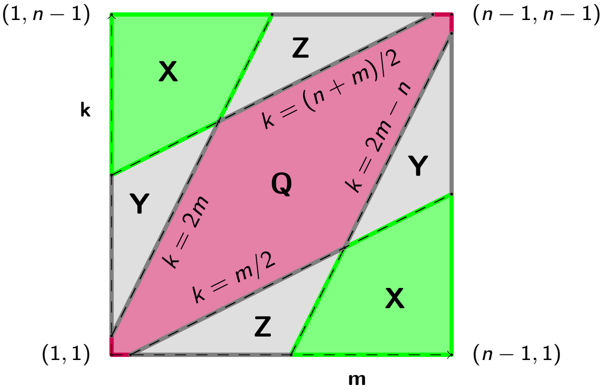
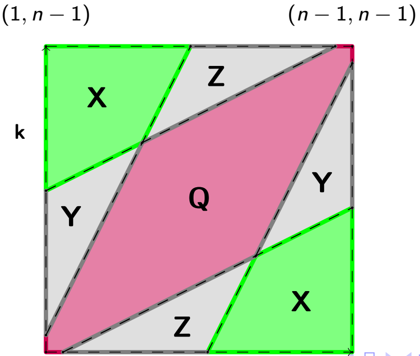


(m,k,j)-projective spaces

collinearly and/or concurrently connected

$(m,k) \in$	Collinearity property	Concurrency property
X	for all j: Yes	for all j : Yes
Y	for all j: Yes	exists j_2': No
Z	exists j_2: No	for all j: Yes
Q	exists $\mathbf{j}$: No	exists $\mathbf{j}'$: No



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Question 4

Is there a $\text{Proj}^{j}_{(m,k)}(V)$ with **exactly one** connectivity property? If yes, under which conditions? (e.g. $G = ABA$ but $G \neq BAB$)

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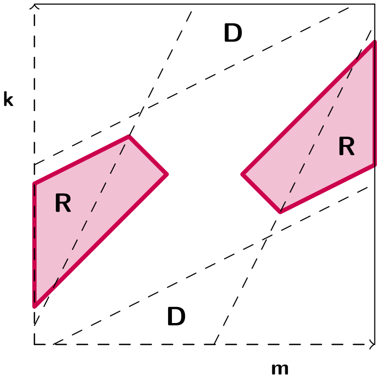

■ $\exists$ possible j s.t., $\mathrm{Proj}^{j}_{(m,k)}(V)$ is collinearly but not concurrently connected.

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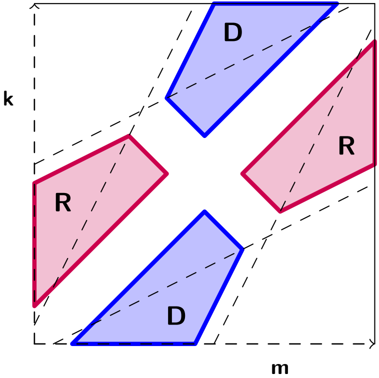

- ∃ possible j s.t., $\mathrm{Proj}^{j}_{(m,k)}(V)$ is collinearly but not concurrently connected.
- ∃ possible j' s.t. $\mathrm{Proj}^{j'}_{(m,k)}(V)$ is concurrently but not collinearly connected

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Thank You

Criteria

Let $A, B < G$, $\alpha := A \in \Omega_A$, and $\beta := B \in \Omega_B$.

- **Geometric Criterion** (Jan Saxl): G-action on Ω_A
  $G = ABA \Leftrightarrow$ the B-orbit α^B intersects nontrivially each G_α-orbit in Ω_A.
  Application:
  (1) [Giudici-James] $S_n = ABA$, A and B conjugate.
  (2) $\mathrm{GL}(V) = ABA$, A: parabolic, B: parabolic/stabiliser of $V = V_1 \oplus V_2$.
- **Restricted Movement Criterion:** G-action on Ω_B
  $G = ABA \Leftrightarrow \Gamma := \beta^A$ has restricted movement:
  $\Gamma^g \cap \Gamma \neq \varnothing$, for all $g \in G$.
  Application:
  (1) $\mathrm{GL}(V) = BAB$, A: parabolic, B: stabiliser of $V = V_1 \oplus V_2$.