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CONJUGACY CLASSES IN FINITE p-GROUPS

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If G is a finite p-group, the sizes of its conjugacy classes are powers of p. This is essentially the only restriction on these sizes, as is seen from

Theorem 1 (J.Cossey - T.O.Hawkes [CH]). *Given any finite set $\mathcal{S}$ of powers of p, including 1, there exists a p-group whose conjugacy class sizes are exactly the members of $\mathcal{S}$.*

The groups constructed by Cossey and Hawkes are of nilpotency class 2.

Problem 1. *Find other constructions, in particular ones that produce groups of higher class.*

Of course, in that problem we have to take into account that the class sizes impose restrictions on the group structure. E.g. if the sizes are $\{1, p\}$, then the nilpotency class has to be 2. More precisely: the class sizes of a p-group G are $\{1, p\}$ iff $|G'| = p$ (Knoche; see also Theorem 3 below). But we can ask, e.g., if, given any set $\mathcal{S} \neq \{1, p\}$ of p-powers, does there exist a group of class 3 whose class sizes are the members of $\mathcal{S}$.

Given an element $x \in G$ whose class size is p^b, we say that $b = b(x)$ is the *breadth* of x. The *breadth* $b(G)$ of G is the maximal breadth of its elements. There is much interest in the relation of this invariant to other invariants of G which measure its departure from commutativity. The following is obvious.

Proposition 2. *If $|G'| = p^k$ and $|G : Z(G)| = p^z$, then $b(G) \leq k$ and $b(G) \leq z - 1$.*

Equality is possible in both inequalities, and one of them has a converse.

Theorem 3 (M.R.Vaughan-Lee [VL]). *If $b(G) = b$ and $|G'| = p^k$, then $k \leq b(b+1)/2$.*

Again equality is possible. There is no bound for $|G : Z(G)|$ in terms of $b(G)$, consider extraspecial groups. But a bound on $|G'|$ imposes a bound on $|G : Z_2(G)|$. For explicit estimates see, e.g., [PS].

It follows from Theorem 3 that the nilpotency class $cl(G)$ is bounded in terms of $b(G)$, but that theorem does not yield the best bound. For a long time many people believed the following

Class - breadth conjecture. *A group of breadth b and class c satisfies $c \leq b + 1$.*

This holds, e.g., if either the breadth is at most $p + 1$, or if the class is at most $p + 3$, or if G is metabelian, and in various other cases. In any case, a linear bound holds.

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Theorem 4 (C.R.Leedham-Green-P.M.Neumann-J.Wiegold [LGNW]).

$$cl(G) \leq \frac{p}{p-1} b(G) + 1.$$

This is proved by a nice counting argument, introducing the important notion of *2-step centralizers*. Let $\gamma_i(G)$ be the ith member of the upper central series of G. The subgroup $C_i(G) = C_G(\gamma_i(G)/\gamma_{i+1}(G))$ is the ith 2-step centralizer of G. There are $c-1$ such centralizers (here $c = cl(G)$), and they are proper subgroups of G. Suppose that $x \notin C_{c-1}(G) = C_G(\gamma_{c-1}(G))$. Since $\gamma_{c-1}(G)/\gamma_c(G) \leq Z(G/\gamma_c(G)) \leq C_{G/\gamma_c(G)}(x\gamma_c(G))$, we obtain $b(x\gamma_c(G)) < b(x)$. Now suppose that $x \notin \cup C_i(G)$. Then looking successively in the factor groups $G/\gamma_i(G)$, we see that $b(x) \geq cl(G)-1$. This implies the class breadth conjecture, provided we can find an appropriate x. If $cl(G) \leq p+1$, then the number of 2-step centralizers is at most p, and since a p-group cannot be the union of p proper subgroups, there is an element x as required. An easy application of the three subgroups lemma shows that $C_1 \leq C_i$, for all i, and therefore C_1 can be omitted from the above considerations. This shows that the class breadth conjecture holds if $cl(G) \leq p+2$, and some elaboration of the argument yields the other cases mentioned above. In the general case we cannot ensure that x exists, but, using the fact that we are dealing with proper subgroups, the authors of [LGNW] estimate the average number of 2-step centralizers containing each element, and from this they estimate the average of $b(x)$ and deduce Theorem 4.

For $p = 2$ the inequality can be improved

Proposition 5 (M.Cartwright [C]). $cl(G) \leq \frac{5}{3} b(G) + 1$.

The class-breadth conjecture was eventually disproved by V.Felsch [F], using computer calculations to construct a counter example of order 2^{34}. Moreover, W.Felsch et al constructed 2-groups in which the difference $c - b$ can be arbitrarily big [FNP]. In these examples c is about $b + \sqrt{b}$. No counter examples for odd primes are known.

Problem 2. *Construct counter examples for odd primes (alternatively, prove that they do not exist).*

The nilpotency class can be bounded under weaker assumptions than in Theorem 4.

Theorem 6 (C.R.vaughan-Lee - J.Wiegold [VLW]). *If G is generated by elements of breadth at most b, then $cl(G) \leq b^2 + 1$.*

The author has improved the bound slightly, to $cl(G) \leq b^2 - b + 1$ (provided that $b > 1$) [M2].

Next, let $s = s(G)$ be the minimal breadth of non-central elements of G. The elements, and classes, of breadth s are called *minimal*, and the difference $d = b - s$ is the *gap* of G. Y.Barnea and I.M.Isaacs conjectured that $cl(G)$ is bounded by a function of d [YB]. This indeed holds.

Theorem 7 (A.Jaikin-Zapirain [JZ1]). $cl(G) \leq 2d^2 + 2d + 3$.

Problem 3. *Can the bounds in Theorems 4 to 7 be significantly improved?*

Substituting $d = 0$ in Theorem 7 yields

Corollary 8 (K.Ishikawa [I]). *If all non-central classes of G have the same size, then $cl(G) \leq 3$.*

Let us go back in history. In the 1950's N.Ito initiated a series of papers discussing finite groups with a small number of conjugacy class sizes. In particular, if all non-central classes have the same size, then Ito showed that $G \cong P \times A$, where P is a p-group and A is abelian [It]. That focuses the problem on p-groups, for which Ito proved the existence of a normal abelian subgroup N such that G/N has exponent p. This was improved by Isaacs [Is1], who showed that actually $exp(G/Z(G)) = p$ (this was reproved later, in ignorance of Isaacs and of each other, by both the author [M1] and L.Verardi [V]). Groups of exponent 2 are abelian, and the ones of exponent 3 have nilpotency class at most 3, and thus Isaacs' result implies that for $p = 2$ the class of G is 2, and for $p = 3$ the class is at most 4, but for larger primes we cannot say much, because groups of prime exponent are quite difficult to understand. Then Ishikawa made a break-through by proving Corollary 8 (which is best possible for odd primes). This prompted the Barnea-Isaacs conjecture. Jaikin's proof of the conjecture is by a highly non-trivial modification of Ishikawa's argument. That argument actually shows that it suffices to assume that G is generated by its minimal elements. The author generalized this to

Theorem 9 (A.Mann [M4]). *Let G be a p-group, and let $M(G)$ be the subgroup that is generated by all the minimal elements. Then $cl(M(G)) \leq 3$.*

The proof is independent of Ishikawa's, and provides a shorter and simpler proof of his result. Moreover, while Corollary 8 deals with a severely restricted class of groups, Theorem 9 states a property of all p-groups. Following the proof of Theorem 9, I formulated several conjectures (these certainly occurred also to other authors). Let the conjugacy classes of G have sizes $n_1 = 1 < n_2 = p^{b_s} < \ldots < n_t = p^{b(G)}$.

Conjecture A. *Let G be a finite p-group, and let the numbers t, n_i be as above. Then there exists a function $f(r)$ such that the subgroup H_r of G generated by the classes of sizes $n_1, \ldots, n_r$ has derived length $dl(H_r)$ at most $f(r)$.*

Conjecture B. *If G is generated by the classes of sizes $n_1, \ldots, n_r$, then $dl(G) \leq f(r)$, for some function of r.*

Note that this is implied by Conjecture A, but is not equivalent to it, because the class sizes in H_r may be different from the class sizes in G of the elements of H_r.

The following is still weaker.

Conjecture C. *The derived length of G is bounded by a function $f(t)$ of t.*

Note also that if $t \geq 3$, we cannot bound $cl(G)$. Consider a non-abelian group containing an abelian maximal group. Then $t = 3$, but there are such groups of arbitrarily high class. One motivation for the conjectures is the fact that the "dual" claims, obtained by replacing class sizes by irreducible character degrees, hold: let N_r be the intersection of the kernels of the irreducible characters of G of the r smallest degrees. Then $dl(G/N_r) \leq r$. This is even true for all soluble groups, with bound $2r$, and conjecturally with much better bounds.

A variation on all three conjectures is obtained by allowing the functions $f(r)$ to depend also on p. Theorem 9, combined with the properties of groups of exponents 2 and 3, and with Theorem 13 below, implies

Proposition 10 ([M3], [M4]). *If $t = 3$ and $p = 2$ then G is metabelian, and if $t = 3$ and $p = 3$ then $dl(G) \leq 4$.*

One more special case is known.

Theorem 11 - (Mann [M5]). *Let G be a finite 2-group, and let H_3 be the subgroup which is generated by the classes of size n_2 and n_3. Then $cl(H_3 \cap G^2) \leq 3$ and $dl(H_3) \leq 3$.*

Corollary 12. *Let G be a 2-group in which $t = 4$. Then $dl(G) \leq 3$.*

There are examples of groups with $t = 4$ and derived length 3, but these constructions are for $p \geq 5$ [IM].

One of the difficulties in proving the conjectures and related results is that induction is often not available, because the number t can increase when we move from G to a subgroup or a factor group. The key to proving Theorem 9 was concentrating on the breadth of one element, rather than of the full group. Take an element $x \in G$. Since $G_G(x) \leq C_G(x^p)$, we have $b(x^p) \leq b(x)$, and it is to be expected the usually the inequality is strict. Of course, this need not always be the case. If memory serves, I have heard from K.Harada, discussing the classification of the finite simple groups, the dictum: concentrate all the bad things in one place. Thus we make the

Definition. The *centralizer equality subgroup D(G)* of G is given by

$$D(G) = \langle x \mid x \in G,\ C_G(x^p) = C_G(x) \rangle.$$

Theorem 13 (Mann [M3]). *The centralizer equality subgroup is abelian.*

This is rather surprising, because we do not expect distinct elements with the defining property of $D(G)$ to be related to each other. Nevertheless, the proof, which was suggested by an argument in [Is1], is quite simple.

Proof. Suppose that $D(G)$ is not abelian. Then there exists an element $z \in Z_2(D(G)) - Z(D(G))$, $z^p \in Z(D(G))$. Let x be one of the defining elements of $D(G)$, and write $H = \langle x, z \rangle$. Then $cl(H) \leq 2$, implying $[x^p, z] = [x, z^p] = 1$, and thus $z \in C_G(x^p) = C_G(x)$. Therefore z commutes with all the defining elements of $D(G)$, i.e. $z \in Z(D(G))$, a contradiction.

Corollary 14. *With the notations of Conjecture A, G contains a normal abelian subgroup D such that $exp(G/D) \leq p^{t-1}$.*

Proof. For $x \in G$, among the $t + 1$ elements $x, x^p, ..., x^{p^t}$ there must be two with the same class size, and therefore the same centralizer. If these elements are x^{p^i} and $x^{p^{i+1}}$, then $i \leq t - 1$ and $x^{p^i} \in D(G)$.

For $p = 2$, it is possible to show that $exp(G/D) \leq 2^{t-2}$.

Another breadth diminishing device is given in the following

Proposition 15. *Let A be a normal abelian subgroup of the finite group G, let $z \in A$ and $x \in G$. If $x \notin Z(G)$, then the conjugacy class of $[x, z]$ has size smaller than that of the class of x.*

Proof. Our original proof, by induction, applied only to p-groups. The present proof is due to Isaacs [Is2] and it applies to all finite groups. First, induction shows that we may assume that $G = AC_G(x)$. Then $[A, x] \lhd G$, and $|[A, x]| > 1$ is the size of the conjugacy class of x. Since $[x, z] \in [A, x]$, all the conjugates of $[x, z]$ lie in $[A,x]$, but they do not exhaust that subgroup, hence their number is less than $|[A,x]|$.

Proof of Theorem 9. Write $N = M(G)$, and let A be maximal among the normal abelian subgroups of G that are contained in N. Then $C_N(A) = A$. If x is a minimal element, the last proposition shows that $[A,x] \leq Z(G)$. Since the minimal elements generate N, it follows that $A \leq Z_2(N)$. Then $N' \leq C_N(A) = A$, and thus $N' \leq Z_2(N)$, implying $cl(N) \leq 3$.

If $p = 2$, a separate argument shows that $cl(M(G)) \leq 2$, but for odd primes the class can be 3.

Since Proposition 15 holds for all finite groups, the conclusion of Theorem 9 holds also for many groups that are not necessarily p-groups, e.g. for supersoluble groups. For these results, see [Is2] and [M6].

Proof of Proposition 10 (sketch). The product $N = D(G)M(G)$ has nilpotency class at most 4, and G/N has exponent p. This, combined with properties of groups of exponent 2 or 3, shows that there is a bound on the derived length of the groups mentioned in the proposition. To obtain the exact bounds there, we show that $D(G)$ centralizes $M(G)$, hence $cl(N) \leq 3$. This suffices for $p = 3$, while if $p = 2$ a little more argument is necessary.

There is a variation on Proposition 15.

Proposition 16. *Under the assumptions and notations of Proposition 15, let $y = [z, x, ..., x]$, with k occurrences of x. If $b(x) \geq k$, then $b(y) \leq b(x) - k$.*

Recall the notion of the *gap* of G, that was defined above, preceding Theorem 7. Proposition 16 implies that if the gap equals d, then the elements of A become so called *right (d+1)-Engel elements* in $G/Z(G)$. Consider the case $d = 1$. For odd primes, right 2-Engel elements lie in $Z_3(G)$. It follows that $A \leq Z_4(G)$. If we take A to be a maximal normal abelian subgroup of G, then $\gamma_4(G) \leq C_G(A) = A \leq Z_4(G)$, implying $cl(G) \leq 7$. This recaptures a special case of Theorem 7, and it is intriguing that this argument, which is very different from the one proving Theorem 7, produces the same bound.

We conclude by mentioning still one more type of results. Using the notations preceding Conjecture A, let there be m_i classes of size n_i. The nature of these numbers is far from clear. Obviously, $m_1 = |Z(G)|$ is a power of p. The other m_i's are multiples of $p-1$, and the papers [Mc],[LMM],[M3],[JZNO] discuss the possibility of equality $m_i = p - 1$ for some i. One major result is

Theorem 17 - (Jaikin-Zapirain [JZ2]). *Given a number A, there are only finitely many p-groups for which $m_i \leq A$, for all i.*

This was extended to all finite soluble groups [JZ3], and conjecturally it holds for all finite groups, see [Ng]. The corresponding result for character degrees holds [Cr].

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