On Oliver's p-group conjecture

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Motivation

Let p be a prime, S a finite p-group and $\mathcal{F}$ a (saturated) fusion system on S.

The Martino-Priddy conjecture:

- Is there a p-local finite group $(\mathcal{L}, \mathcal{F}, S)$? – YES!
- If 'yes', is $(\mathcal{L}, \mathcal{F}, S)$ unique? – ...essentially YES!

(For the concepts, see the survey article by Broto, Levi, Oliver.)

R. Oliver: Suppose p odd. The Martino-Priddy conjecture holds if $J(S) \leq \mathfrak{X}(S)$ for any finite p-group S, where $J(S)$ is the *Thompson subgroup*, generated by all elementary abelian subgroups of S of maximal order, and $\mathfrak{X}(S)$ is... a certain characteristic subgroup of S (later coined *Oliver subgroup*).

The conjecture

Conjecture (Oliver)

Suppose p odd. Then $J(S) \leq \mathfrak{X}(S)$ for any finite p-group S.

R. Oliver proved it whenever S 'occurs' in a fusion system $\mathcal{F}$ realised by a finite group by reducing to finite simple groups and using the CFSG. The question for *exotic* fusion systems is still unsolved.

Thus, the Martino-Priddy conjecture is true whenever $\mathcal{F}$ is not exotic. But what if $\mathcal{F}$ is exotic?

Moderation

Remark

The condition

$$J(S) \leq \mathfrak{X}(S) \quad \text{for any finite } p\text{-group } S$$

is sufficient but not necessary. A milder sufficient condition: $\mathfrak{X}(S)$ contains a *universally weakly closed* subgroup of S , for any S.

$Q \leq S$ is universally weakly closed in S if for each $S' \geq S$ and each fusion system $\mathcal{F}$ on S' such that S is strongly $\mathcal{F}$-closed in S', then Q is weakly $\mathcal{F}$-closed in S'.

Definitions

Let p be a prime and S a finite p-group.

- $E \leq G$ is *elementary abelian* (or *el. ab.* for short) if E is abelian of exponent p. The *rank* of E is $\mathrm{rk}(E) = \dim_{\mathbb{F}_p} E$. The rank of S is $\mathrm{rk}(S) = \max_{E \in \mathcal{E}(S)} \mathrm{rk}(E)$.
- $\mathcal{E}(G) = \{E \leq G \mid E \quad \text{el. ab.} : \quad E \neq 1\}$ is the set of non-trivial el. ab. subgroups of G.
- $J(S) = \langle E \in \mathcal{E}(S) \mid \mathrm{rk}(E) = \mathrm{rk}(S) \rangle$.
- $\Omega_1(S) = \langle x \in S \mid x^p = 1 \rangle$
- Commutators: $[x, y; 1] = [x, y] = x^{-1}y^{-1}xy$, and $[x, y; n] = [[x, y; n-1], y]$ for $n \geq 2$.

Oliver subgroup

Henceforth, p is an odd prime and S a finite p-group.
$R \leq S$ has a *Q-series* if $R \trianglelefteq S$ and there exist subgroups $Q_0, \ldots, Q_n \trianglelefteq S$ with

$$1 = Q_0 \quad , \quad R = Q_n \quad \text{and}$$

$$[\Omega_1(C_S(Q_{i-1})), Q_i; p-1] = 1 \; \forall \; 1 \leq i \leq n.$$

The *Oliver subgroup* of S is the largest subgroup $\mathfrak{X}(S)$ of S which has a Q-series.

Preliminaries

1. $\mathfrak{X}(S)$ is well-defined: if R_1, R_2 have Q-series, then $R_1 R_2$ too.
2. $\mathfrak{X}(S)$ is centric ($C_S(\mathfrak{X}(S)) = Z(\mathfrak{X}(S))$) and characteristic in S.
3. For $p = 2$ we get $\mathfrak{X}(S) = C_S(\Omega_1(S))$.
4. If $\mathrm{cl}(S) < p - 1$ or if $\mathrm{rk}(Z(\mathfrak{X}(S))) < p$, then $\mathfrak{X}(S) = S$.
5. $\mathfrak{X}(S) \geq A$ for any normal abelian subgroup A of S.
6. If $Q \trianglelefteq S$ and $[\Omega_1(Z(\mathfrak{X}(S))), Q; p-1] = 1$ then $Q \leq \mathfrak{X}(S)$.

An example

Let $S = C_p \wr C_p$.
Recall $|S| = p^{p+1}$ and $\mathrm{cl}(S) = p$.

$$\mathfrak{X}(S) = J(S) \cong \underbrace{C_p \times \cdots \times C_p}_{p \text{ factors}} \quad \text{is the base subgroup of } S.$$

Thus S is the 'smallest' case where $\mathfrak{X}(S) < S$.

Reformulation of Oliver's conjecture

Let (PS) be the property: *Let G be a non-trivial finite p-group and V a (finitely generated) faithful $\mathbb{F}_p G$-module. The restriction* $\mathrm{Res}^G_{\langle z \rangle}(V)$ *has a non-trivial projective direct summand for every* $1 \neq z \in \Omega_1(Z(G))$.

Theorem (Green, Héthelyi, Lilienthal)

Oliver's conjecture is equivalent to: 'any non-trivial finite p-group G has no F-module satisfying (PS).'

F-modules and offenders

Let G, V be as in the above theorem. For $H \leq G$ put

$$j_H(V) = \frac{|H|\,|CV(H)|}{|V|}. \quad \text{Note that } j_1(V) = 1 \quad .$$

H is *quadratic* (on V) if $[V, H, H] = 1$.
$E \in \mathcal{E}(G)$ is an *offender* (for V) if $j_E(V) \geq 1$. E is a *best offender* (for V) if $j_F(V) \leq j_E(V)$ for all $1 \leq F \leq E$. V is an *F-module* (for G) if V has an offender. F-module stands for *failure of Thompson's factorisation*. (See [GLS2])

Consequence of Timmesfeld replacement theorem

V is an F-module iff V has a *quadratic best offender*.

Underlying the recast

The only case to consider: $\mathfrak{X}(S) < S$, i.e. $G := S/\mathfrak{X}(S) > 1$.
If $J(S) \leq \mathfrak{X}(S)$ iff each $E \in \mathcal{E}(S)$ of maximal order lies in $\mathfrak{X}(S)$. The reformulation translates this condition in terms of the faithful representations of G over $\mathbb{F}_p$.
Let V be a faithful $\mathbb{F}_pG$-module. We want to show that if V satisfies (PS), then V is not an F-module, i.e. V has no quadratic best offender.

Remark

Any non-trivial p-group G arises as $S/\mathfrak{X}(S)$ for some S. Indeed, take any faithful $\mathbb{F}_pG$-module V and let $S = V \rtimes G$. Then $V = \mathfrak{X}(S)$.

Outcome

Theorem (Green, Héthelyi, M.)

Oliver's conjecture holds for any finite p-group S such that $G = S/\mathfrak{X}(S)$ satisfies one of the following conditions.

1. $\mathrm{cl}(G) \leq 4$.
2. *G is metabelian.*
3. $\mathrm{rk}(G) \leq p$.

Corollary

Oliver's conjecture holds for any finite p-group S such that $G = S/\mathfrak{X}(S)$ satisfies one of the following conditions.

1. *G has maximal nilpotence class.*
2. *G is a regular 3-group.*

Stepping stones

Results by Chermak, Delgado, Meierfrankenfeld and Stellmacher enable us to show:

Theorem

Let G be a non-trivial finite p-group and V a faithful $\mathbb{F}_pG$-module.

1. *If $\Omega_1(Z(G))$ has no quadratic elements, then*
   - *if $A \trianglelefteq G$ abelian, then A does not contain any offender.*
   - *If $E \in \mathcal{E}(G)$ is an offender, then $[G', E] \neq 1$.*
2. *If $\Omega_1(Z(G))$ has no quadratic elements and either $\mathrm{cl}(G) \leq 4$ or G is metabelian, then V cannot be an F-module.*
3. *If V satisfies (PS) and $\mathrm{rk}(G) \leq p$, then V cannot be an F-module.*

A 'quadratic' view

As a consequence of Timmesfeld's theorem, all 'reduces' to the analysis of the action of *quadratic* elements and subgroups on faithful $\mathbb{F}_p G$-modules. Hence,
$R \trianglelefteq S$ has a Y*-series* if if $R \trianglelefteq S$ and there exist subgroups $Y_0, \ldots, Y_n \trianglelefteq S$ with

$$1 = Y_0 \quad , \quad R = Y_n \quad \text{and}$$

$$[\Omega_1(C_S(Y_{i-1})), Y_i; 2] = 1 \; \forall \; 1 \leq i \leq n.$$

Let $\mathcal{Y}(S)$ be the largest subgroup of S which has a Y-series.

Comparison

As for $\mathfrak{X}(S)$, the subgroup $\mathcal{Y}(S)$ is

- well-defined and characteristic in S;
- any finite p-group G arises as $S/\mathcal{Y}(S)$ for some S;
- $\mathcal{Y}(S)$ contains every abelian normal subgroup of S, and thus $\mathcal{Y}(S)$ is centric in S.

If $p = 2$, then $\mathcal{Y}(S) = S$ and if $p = 3$ then $\mathcal{Y}(S) = \mathfrak{X}(S)$. Oliver's conjecture holds for S whenever the conditions in the theorem below are satisfied.

Theorem

Let p be an odd prime and G, S finite p-groups with $G = S/\mathcal{Y}(S)$. Then $J(S) \leq \mathcal{Y}(S)$ if and only for every F-module V there are quadratic elements in $\Omega_1(Z(G))$.

Improvements

Theorem

Suppose that G satisfies $\Omega_1(Z(\mathcal{Y}(G))) = \Omega_1(Z(G))$ (this holds if $\mathcal{Y}(G) = G$ and hence if $G = \langle A \trianglelefteq G \mid A \text{ abelian} \rangle$). Then for every F-module V for G there are quadratic elements in $\Omega_1(Z(G))$. In particular, Oliver's conjecture holds.

Theorem

Let P be a Sylow p-subgroup of some general linear group $\mathrm{GL}_n(q)$. Then

- $J(P) \leq \mathcal{Y}(P)$
- *for every F-module V there are quadratic elements in $\Omega_1(Z(P))$.*

A last word

This last result implies that Oliver's conjecture holds for every finite p-group S such that either S, or the factor group $G = S/\mathcal{Y}(S)$, is a Sylow p-subgroup of some $\mathrm{GL}_n(q)$, including the case of the Sylow p-subgroups of the symmetric groups. Recall that these are either generated by their abelian normal subgroups (defining characteristic), or direct products of iterated wreath products of cyclic p-groups (non-defining characteristic).