Poset of elementary abelian subgroups of finite p-groups

partly joint with George Glauberman

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Contents

The basics

- p a prime;
- G a finite p-group.
- $E \leq G$ is *elementary abelian* (or *el. ab.* for short) if E is abelian of exponent p. The *rank* of E is $\mathrm{rk}(E) = \dim_{\mathbb{F}_p} E$.
- $\mathcal{E}(G) = \{E \leq G \mid E \quad \text{el. ab.} : \quad \mathrm{rk}(E) \geq 2\}$ is the *poset* of el. ab. subgroups of G of rank at least 2. The order is the *inclusion* $\subseteq$.
- $E, F \in \mathcal{E}(G)$ are *connected* if $\exists\, E_1, \ldots, E_t \in \mathcal{E}(G)$ with

$$E = E_1 \quad , \quad F = E_t \quad \text{and}$$

$$E_i \subseteq E_{i+1} \quad \text{or} \quad E_{i+1} \subseteq E_i \quad \text{for all } 1 \leq i.$$

In this case, write $E \sim F$.

Examples

1. G cyclic $\Rightarrow$ $\mathcal{E}(G) = \emptyset$.
2. G abelian $\Rightarrow$ $\mathcal{E}(G)$ is connected.
3. p odd and $G = p_+^{1+2}$ $\Rightarrow$ $\mathcal{E}(G)$ is a set of $p+1$ points.
4. p odd and $G = C_p \wr C_p$ $\Rightarrow$ $\mathcal{E}(G)$ is a *big connected component* disjoint union with p^{p-2} *isolated* points, all G conjugate.

Our conventions

Henceforth:

p is an *odd* prime

and

A *component* of $\mathcal{E}(G)$ = G-conjugacy class of connected component of $\mathcal{E}(G)$.

$\#\mathcal{E}(G)$ = number of components of $\mathcal{E}(G)$.

Key lemma

$\mathcal{E}(G)$ is the disjoint union of a *big component* and isolated vertices: Suppose G is not cyclic. Then,

- $\exists\ E_0 \in \mathcal{E}(G)$ with $C_p^2 \cong E_0 \trianglelefteq G$;
- $E \in \mathcal{E}(G)$ with $\mathrm{rk}(E) > 2 \ \Rightarrow\ E \sim E_0$.

In particular,

$\#\mathcal{E}(G) > 1 \quad \Leftrightarrow \quad G$ has maximal el. al. subgroups of rank 2.

Hunch & . . . ?

Most finite p-groups G have $\#\mathcal{E}(G) = 1$.
What can be said of G if $\#\mathcal{E}(G) > 1$?

Known facts

Suppose $\#\mathcal{E}(G) > 1$ and let $E, E_0 \in \mathcal{E}(G)$ with E maximal el. ab. of rank 2 and $E_0 \trianglelefteq G$ also of rank 2. Then:

1. $Z(G)$ is cyclic.
2. $C_G(E) = E * L$ with L cyclic.
3. $N_G(E) = EE_0 * L$ with $EE_0 \cong p_+^{1+2}$. Thus $|N_G(E) : C_G(E)| = p$.
4. If $E \ntrianglelefteq G$ then $\mathrm{cl}(G) = \log_p\left(\frac{|G|}{|L|}\right)$. Furthermore, E_0 char G is the unique el. ab. of rank 2 normal in G.
5. If $E \trianglelefteq G$ then $G = EE_0 * L$. In particular, $\mathrm{rk}(G) = 2$ and $\#\mathcal{E}(G) = p + 1$.

Two questions and answers

Let G be a finite p-group, p odd. Is there n such that:

1. if $\mathrm{rk}(G) > n$, then $\#\mathcal{E}(G) = 1$?
2. $\#\mathcal{E}(G) < n$?

Answers

For any odd prime p and any finite p-group G:

1. if $\mathrm{rk}(G) > p$, then $\#\mathcal{E}(G) = 1$.
2. $\#\mathcal{E}(G) < p + 2$.

Th answer to question 1 has been obtained in collaboration with G. Glauberman.

Bounding rk(G)

Suppose that G is a finite p-group of rank > 2.

Linear algebra note:

if $V \trianglelefteq G$ is el. ab. and $\mathrm{rk}(V) > p$ then $\#\mathcal{E}(G) = 1$.

Improvement using Lazard correspondence:

if $V \trianglelefteq G$ is of exponent p, $\mathrm{cl}(V) < p$, and $|V| > p^p$ then $\#\mathcal{E}(G) = 1$.

Group theory notes:

1. $\#\mathcal{E}(G) = 1 \iff \mathrm{rk}(C_G(x)) > 2$, $\forall\, x \in G$ with $x^p = 1$;
2. if $p = 3$ and $\mathrm{rk}(G) > 3$ then $\#\mathcal{E}(G) = 1$. (Jonah-Konvisser)
3. if $p \geq 5$ and $A \in \mathcal{E}(G)$ with $\mathrm{rk}(A) = n$ then $\exists\ B \in \mathcal{E}(G)$ with $\mathrm{rk}(B) = n$ and $B \trianglelefteq \langle B^G \rangle$. (Alperin-Glauberman)

About the proof

We want to show that if $p > 3$, if $\#\mathcal{E}(G) > 1$, and if E is a non-normal maximal el. ab. subgroup of G, then $\mathrm{rk}(G) \leq p$.

Strategy

Proceed by contradiction: Suppose $\exists\ A \in \mathcal{E}(G)$ with $\mathrm{rk}(A) = p + 1$ and $A \lhd \langle A^G \rangle =: N$.
Choose normal subgroups of G of class $< p$ and exponent p.
Thus, by 'Lazard' remark, these have order $\leq p^p$; namely:

$$M = \Omega_1(Z_{p-1}(N)) \quad Y = \Omega_1(Z_2(N)) \quad W = \Omega_1(Z(N))$$

About the proof

Either $Y \leq A$ or $Y \nleq A$.
If $Y \leq A$ then $Y \leq Z(N)$ (as any normal abelian subgroup of G contained in A) $\Rightarrow$ $A \cap Z_2(N) = A \cap Z(N)$ $\Rightarrow$ $A \leq M$.
Contradiction: $|A| = p^{p+1} > |M|$.
If $Y \nleq A$, then (…exercise…)
$A \cap Z_2(N) = A \cap Z_3(N)$ $\Rightarrow$ $A \leq M$. Contradiction:
$|A| = p^{p+1} > |M|$.

Remark

For $p = 2$, there is a 2-group of rank 4 with a non-normal maximal Klein four group. See article [GM2010] for details.

Köszönöm Laci!

Let G be a finite p-group. A subgroup A of G is *soft* if $A = C_G(A)$ and $|N_G(A) : A| = p$. A is *deep soft* if A is soft and $A \ntrianglelefteq G$, i.e. $|G : A| > p$.

L. Héthelyi proved:

Theorem

- *Every soft subgroup of G is contained in a unique maximal subgroup.*
- *Any two soft subgroups contained in the same maximal subgroup of G are conjugate.*
- *Let A be a deep soft subgroup of G, and put $H = G'Z(N_G(A))$. Then, $G/H \cong C_p \times C_p$ and H is independent of A.*

Un jeu d'enfant

- We want to prove that $\#\mathcal{E}(G) < p + 2$.
- By 'Known Facts', we may assume that G contains a non-normal maximal el. ab. subgroup E of rank 2. Hence $A = C_G(E) = E * L$ is deep soft in G and L is cyclic.
- Héthelyi's theorem $\Rightarrow$ there are at most $p + 1$ maximal subgroups of G which may contain soft subgroups, each of these may contain at most one conjugacy class of soft subgroups, i.e. an 'isolated component' of $\mathcal{E}(G)$.
- $H < C_G(E_0) < G$ for the unique $E_0 \in \mathcal{E}(G)$ with $E_0 \lhd G$ and $\mathrm{rk}(E_0) = 2$.
- Add the numbers to get the maximal number of components: 1 big and p isolated. $\Rightarrow$ QED

Class-breadth conjecture

Let G be a finite p-group and $x \in G$. The *breadth of* x is

$$b(x) = \log_p |G : C_G(x)|. \quad \text{Thus} \quad b(G) = \max_{x \in G} b(x)$$

is the *breadth of* G.

Conjecture (Leedham-Green, Neumann, Wiegold,1969)

Let G be a finite p-group. Then $\mathrm{cl}(G) \leq b(G) + 1$.

The conjecture fails for $p = 2$ but is still an open question for p odd. In view of our results:

Corollary

If G is a finite p-group (p odd) which has a non-normal maximal el. ab. subgroup of rank 2, then $\mathrm{cl}(G) \leq b(G) + 1$.

Endotrivial modules

Let $k = \overline{k}$ be a field of characteristic p and G a finite group of order divisible by p. A finitely generated kG-module M is *endotrivial* if $\operatorname{End}_k M \cong k$ in the stable module category $\operatorname{stmod}(kG)$. The *group of endotrivial kG-modules* is

$$T(G) = \{ \text{ iso. classes of endotrivial } kG\text{-modules in } \operatorname{stmod}(kG)\}$$

Known: $T(G)$ is a finitely generated abelian group. Its torsion-free rank is $\#\mathcal{E}(G)$, i.e. the G-conjugacy classes of connected components of $\mathcal{E}(P)$ for a Sylow p-subgroup P of G.

Corollary

For any odd p and finite group G the group $T(G)$ has torsion-free rank at most $p+1$. Furtermore, if the p-rank of G is greater than p, then $T(G)$ has torsion-free rank 1.